On Fuzzy Ideals Of Subtraction Semigroups

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Abstract: In this paper, we introduce the notion of fuzzy interior ideal, fuzzy bi-ideal, intuitionistic fuzzy interior ideal and intuitionistic fuzzy bi-ideal of a subtraction semigroup. We characterize a non-empty subset of a subtraction semigroup X through intuitionistic fuzzy ideal, intuitionistic fuzzy bi-ideal and intuitionistic fuzzy interior ideal. We give some equivalent conditions related to these notions.

Key words: Subtraction semigroups, fuzzy interior ideal, fuzzy bi-ideal, intuitionistic fuzzy interior ideal, intuitionistic fuzzy bi-ideal.

AMS Mathematics Subject Classification (2000): 20M12, 03E72, 03F55, 03G25, 06B10, 06D99

1. Introduction and Preliminaries

B. M. Schein ([11]) considered systems of the form \((\Phi, \circ, -)\), where \(\Phi\) is a set of functions closed under the composition "\(\circ\" of functions (and hence \((\Phi, \circ, -)\) is a function semigroup) and the set theoretic subtraction "\(-\" (and hence \((\Phi, \circ, -)\) is a subtraction algebra in the sense of [3]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka ([14]) discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. B. Jun, H. S. Kim and E. H. Roh ([6]) introduced the notion of ideals in subtraction algebras and discussed characterizations of ideals. In [7], Y. B. Jun and H. S. Kim established the ideal generated by a set and discussed related results.

After the introduction of fuzzy sets by Zadeh ([13]), several researchers were conducted on the generalizations of the notion of fuzzy set. The concept of intuitionistic fuzzy set was introduced by Atanassov [1,2] as a generalization of the notion of fuzzy set. In [12], Williams introduce a notion of fuzzy ideals in near-subtraction semigroups, and studied their related properties. In [5], Dheena and Mohanraj introduced the notion of fuzzy ideal, fuzzy weak ideal, fuzzy weakly prime left ideal and fuzzy prime left ideal system of a near-subtraction semigroup and discussed some related results.
In this paper, we introduce the concepts of fuzzy interior ideal, fuzzy bi-ideal, intuitionistic fuzzy interior ideal and intuitionistic fuzzy bi-ideal in a subtraction semigroup and give some related properties.

Definition 1.1. An algebra \((X;\cdot)\) with a single binary operation "\(-\)" is called a subtraction algebra if for all \(x,y,z \in X\) the following conditions hold:

1. \(x-(y-x)=x\),
2. \(x-(x-y)=y-(y-x)\),
3. \((x-y)-z=(x-z)-y\).

The subtraction determines an order relation on \(X\) : \(a \leq b \iff a-b=0\), where \(0=a-a\) is an element that doesn't depend on the choice of \(a \in X\).

In a subtraction algebra, the following are true [6,7]:

1. \((x-y)-y=x-y\),
2. \(x-0=x\) and \(0-x=0\),
3. \((x-y)-x=0\),
4. \((x-y)-(y-x)=x-y\),
5. \(x \leq y\) implies \(x-z \leq y-z\) and \(z-y \leq z-x\) for all \(z \in X\).

Definition 1.2. A non-empty subset \(S\) of a subtraction algebra \(X\) is said to be a subalgebra of \(X\) if \(x-(y) \in S\) whenever \(x,y \in S\).

Definition 1.3. Let \(X\) be a semigroup. By a subsemigroup of \(X\), we mean a non-empty subset \(S\) of \(X\) such that \(xy \in S\) for all \(x,y \in S\).

Definition 1.4. A semigroup \(X\) is said to be regular if, for each \(a \in X\), there exists an \(x \in X\) such that \(a=axa\).

Definition 1.5. A non-empty set \(X\) together with the binary operations "\(-\)" and "." is said to be a subtraction semigroup if it satisfies the following properties:

1. \((X;\cdot)\) is a subtraction algebra,
2. \((X,\cdot)\) is a semigroup,
3. \(x(y-z)=xy-xz\) and \((x-y)z=xz-yz\) for all \(x,y,z \in X\).

It is clear that, in a subtraction semigroup \(X\), we have \(0x=0\) and \(x0=0\) for all \(x \in X\).

Definition 1.6. \((X;\cdot,\cdot)\) be a subtraction semigroup. A non-empty subset \(I\) of \(X\) is called

1. \((I\subseteq X)\) a left ideal if \(I\) is a subalgebra of \((X;\cdot)\) and \(xi \in I\) for all \(x \in X\) and \(i \in I\),
2. \((I\subseteq X)\) a right ideal if \(I\) is a subalgebra of \((X,\cdot)\) and \(ix \in I\) for all \(x \in X\) and \(i \in I\),
3. \((I\subseteq X)\) an ideal if \(I\) is both a left and a right ideal.

Example 1.7. Let \((X;\cdot,\cdot)\) be a subtraction semigroup and \(a \in X\).

1. \(A_a=\{x \in X: ax=0\}\) is a right ideal of \(X\).
(ii) $aX = \{ax : x \in X\}$ is a right ideal of $X$.

**Definition 1.8.** Let $X$ be a non-empty set. A mapping $\mu : X \rightarrow [0,1]$ is called a fuzzy set of $X$.

**Definition 1.9.** The level set of a fuzzy set $\mu$ of $X$ is defined as $U = U(\mu; t) = \{x \in X : \mu(x) \geq t\}$ for all $0 \leq t \leq 1$.

**Definition 1.10.** An intuitionistic fuzzy set (IFS) $A$ in a non-empty set $X$ is an object having the form

$$A = \{ (x, \mu(x), \gamma(x)) : x \in X\}$$

where the functions $\mu : X \rightarrow [0,1]$ and $\gamma : X \rightarrow [0,1]$ denote the degree of membership and the degree of nonmembership, respectively, and

$$0 \leq \mu(x) + \gamma(x) \leq 1$$

for all $x \in X$. For the sake of simplicity, we shall use the symbol $A = (\mu, \gamma)$ for the IFS $A = \{ (x, \mu(x), \gamma(x)) : x \in X\}$.

In what follows, let $X$ be a subtraction semigroup, unless otherwise specified.

### 2. Fuzzy Interior Ideals and Fuzzy Bi-ideals

**Definition 2.1.**

(i) A subset $Y$ of $X$ is called an interior ideal of $X$ if $Y$ is a subalgebra of $X$ and $XYX \subseteq Y$.

(ii) A subset $Y$ of $X$ is called a bi-ideal of $X$ if $Y$ is a subalgebra of $X$ and $YXY \subseteq Y$.

**Example 2.2.** Let $X = \{0, a, b, c\}$ in which "-" and "." are defined by the following table:

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Then it is easily seen that $(X; -,. )$ is a subtraction semigroup and $I = \{0, b\}$ is both an interior ideal and a bi-ideal of $X$. But $I$ is not an ideal of $X$.

**Theorem 2.3.**

(i) Every ideal of $X$ is an interior ideal.

(ii) If $X$ contains an identity element then every interior ideal of $X$ is an ideal.

(iii) Any left (right, two-sided) ideal of $X$ is a bi-ideal.
Proof. It is clear by the definitions.

If I is a single side ideal of a subtraction semigroup X then I is not an interior ideal of X. For example, the sets Aₐ and aX in Example 1.7 are not interior ideals generally.

**Definition 2.4.** For a fuzzy set µ in X, consider the following axioms:

(i) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$,
(ii) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$

for all $x, y \in X$. Then µ is called a fuzzy subalgebra of X if it satisfies (i), and µ is called a fuzzy subsemigroup of X if it satisfies (ii).

Let $A$ be a non-empty subset of X and $\mu_A$ be a fuzzy set in X defined by

$$\mu_A(x) = \begin{cases} s, & \text{if } x \in A \\ t, & \text{if } x \notin A \end{cases}$$

(2.1)

for all $x \in X$ and $s, t \in [0, 1]$ with $s > t$.

**Theorem 2.5.** Let $A$ be a non-empty subset of X and $\mu_A$ be a fuzzy set in X defined in (2.1). Then

(i) $A$ is a subalgebra of X if and only if $\mu_A$ is a fuzzy subalgebra of X

(ii) $A$ is a subsemigroup of X if and only if $\mu_A$ is a fuzzy subsemigroup of X.

Proof. (i) Let $x, y \in X$. If $x, y \in A$, then $\mu_A(x - y) = s = \mu_A(x) = \mu_A(y)$ since $x - y \in A$. If $x \notin A$ or $y \notin A$, then $\mu_A(x) = t$ or $\mu_A(y) = t$. Hence $\mu_A(x - y) \geq t = \min\{\mu_A(x), \mu_A(y)\}$ by the definition of $\mu_A$. Hence $\mu_A$ is a fuzzy subalgebra of X. Conversely, let $x, y \in A$. Then since $\mu_A(x - y) \geq \min\{\mu_A(x), \mu_A(y)\} = s$, we have $\mu_A(x - y) = s$ and so $x - y \in A$.

(ii) Similar to the proof of (i).

**Definition 2.6.** [12] A fuzzy set $\mu$ in X is called a fuzzy ideal of X if it satisfies the following axioms:

(FI1) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$,
(FI2) $\mu(xy) \geq \mu(y)$,
(FI3) $\mu(xy) \geq \mu(x)$

for all $x, y \in X$.

Note that $\mu$ is called a fuzzy left ideal of X if it satisfies (FI1) and (FI2), and $\mu$ is called a fuzzy right ideal of X if it satisfies (FI1) and (FI3).
Example 2.7. Let $X=\{0,a,b,c\}$ in which "-" and "." are defined by the following table:

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Then $(X;\cdot,\cdot)$ is a subtraction semigroup ([4]). Let $\mu$ be a fuzzy set on $X$ defined by $\mu(0)=0.8$, $\mu(a)=0.5$, $\mu(b)=0.3$, $\mu(c)=0.1$. Then it is easy to see that $\mu$ is a fuzzy ideal of $X$.

Proposition 2.8. Let $\mu$ be a fuzzy ideal in $X$. Then $\mu(0)\geq \mu(x)$ for all $x\in X$.

Proof. Using (FI2), we have $\mu(0)=\mu(0x)\geq \mu(x)$ for all $x\in X$.

For the sake of completeness, we give the following theorems which are special cases of the Theorem 3.3 and Theorem 3.4 in [12].

Theorem 2.9. Let $\mu$ be a fuzzy left (right) ideal in $X$. Then the set $I_\mu=\{x\in X : \mu(x)=\mu(0)\}$ is a left (right) ideal of $X$.

Theorem 2.10. The subset $A$ of $X$ is an ideal of $X$ if and only if $A_\mu$ defined by in (2.1), is a fuzzy ideal of $X$. Moreover $I_{A_\mu}=A$.

Corollary 2.11. Let $\mu$ be a fuzzy subset in $X$. If a non-empty level subset $U$ of $\mu$ is an ideal of $X$, then $U_\mu$ is a fuzzy ideal of $X$.

Theorem 2.12. [5] Let $\mu$ be a fuzzy subset in $X$. $\mu$ is a fuzzy ideal of $X$ if and only if any non-empty level subset $U$ of $\mu$ is an ideal of $X$.

Definition 2.13. For a fuzzy set $\mu$ in $X$, consider the following axiom:

(FII2) $\mu(xay)\geq \mu(a)$

for all $x,y,a\in X$. $\mu$ is called a fuzzy interior ideal (FII) of $X$ if it satisfies FI1 and FII2.

Proposition 2.14. Every FI of $X$ is a FII.

Proof. By (FI2) and (FI3), we have $\mu(xay)=\mu((xa)y)\geq \mu(xa)\geq \mu(a)$ for all $x,y,a\in X$.

Theorem 2.15. $A$ is an interior ideal of $X$ if and only if $A_\mu$ is a fuzzy interior ideal of $X$.

Proof. Let $A$ be an interior ideal of $X$ and denote $A_\mu=\mu$. Let $x$, $a$, $y\in X$. If $a\in A$, we have $xay\in A$ since $A$ is interior ideal of $X$. So we get $\mu(xay)=s=\mu(a)$. If $a\not\in A$, then by the definition of $\mu$, $\mu(xay)\geq t=\mu(a)$. Since $A$ is also a subalgebra of $X$, by Theorem 2.5 (i), $\mu$
satisfies (FI1). Conversely, let $\mu$ is a FII of $X$. By Theorem 2.5 (i), $A$ is a subalgebra. Now let $xay$ be any element of $XAX$. Then since $a \in A$ and $\mu$ is FII of $X$, we have $\mu(xay) \geq \mu(a) = s$ and hence we get $\mu(xay) = s$. So it is obtained that $xay \in A$.

**Theorem 2.16.** Let $\mu$ be a fuzzy set of $X$. If $\mu$ is a FII of $X$, then each non-empty level set $U = U(\mu; t)$ $(0 \leq t \leq 1)$ of $X$ is an interior ideal of $X$.

**Proof.** Let $\mu$ be a fuzzy interior ideal of $X$ and $U = U(\mu; t) = \{x \in X : \mu(x) \geq t\}$ for any $t$ where $0 \leq t \leq 1$ be a level set of $\mu$. For $x, y \in U$, since $\mu(x) \geq t$ and $\mu(y) \geq t$ and $\mu$ satisfies (FI1), we get $\mu(x-y) \geq \min\{\mu(x), \mu(y)\} \geq t$. Hence $U$ is a subalgebra of $X$. For all $x, z \in X$ and $y \in U$ we have $\mu(xyz) \geq \min\{\mu(x), \mu(z)\}$. So we obtain $xyz \in U$, that is, $XUX \subseteq U$.

**Definition 2.17.** A fuzzy set $\mu$ in $X$ is called a fuzzy bi-ideal (FBI) of $X$ if it satisfies (FI1) and the following condition:

$$\text{(BI2)} \quad \mu(xyz) \geq \min\{\mu(x), \mu(z)\}$$

for all $x, y, z \in X$.

**Proposition 2.18.** Every FI of $X$ is an FBI of $X$.

**Proof.** Let $\mu$ be a FI of $X$. Then we get $\mu(xyz) = \mu((xy)z) \geq \mu(z)$ and $\mu(xyz) = \mu(x(yz)) \geq \mu(x)$. Hence we can write $\mu(xyz) \geq \min\{\mu(x), \mu(z)\}$ for all $x, y, z \in X$.

**Theorem 2.19.** Let $Y$ be a non-empty subset of $X$ and $\mu_Y$ be a fuzzy set of $X$ defined by (2.1). Then $Y$ is a bi-ideal of $X$ if and only if $\mu_Y$ is a FBI of $X$.

**Proof.** Denote $\mu_Y = \mu$. Let $x, w, y \in X$. If $x, y \in Y$, we have $xwy \in Y$ since $Y$ is a bi-ideal. So we get $\mu(xwy) = s = \mu(x) = \mu(y) = \min\{\mu(x), \mu(y)\}$. If $x \notin Y$ or $y \notin Y$, then by the definition of $\mu$, we have $\mu(xwy) \geq \mu(x) = \min\{\mu(x), \mu(y)\}$ (or $\mu(xwy) \geq \mu(y) = \min\{\mu(x), \mu(y)\}$). Since $Y$ is also a subalgebra of $X$, by Theorem 2.5 (i), $\mu$ satisfies (FI1). Conversely, let $\mu$ is a FBI of $X$. By Theorem 2.5 (i), $Y$ is a subalgebra. For all $x, y \in Y$ and $w \in X$, we have $\mu(xwy) \geq \min\{\mu(x), \mu(y)\} = s$ and hence we get $\mu(xwy) = s$. So it is obtained that $xwy \in Y$, that is, $YXY \subseteq Y$.

**Theorem 2.20.** Let $\mu$ be a fuzzy set of $X$. If $\mu$ is a FBI of $X$, then each non-empty level set $U = U(\mu; t)$ $(0 \leq t \leq 1)$ of $X$ is a bi-ideal of $X$.

**Proof.** Let $\mu$ be a FBI of $X$ and $U = U(\mu; t) = \{x \in X : \mu(x) \geq t\}$ for any $t$ where $0 \leq t \leq 1$ be a level set of $\mu$. We have that $U$ is a subalgebra of $X$ as in the proof of Theorem 2.16. For all $x, z \in U$ and $y \in X$, we have $\mu(xyz) \geq \min\{\mu(x), \mu(z)\} \geq t$. So we obtain $xyz \in U$, that is, $UXU \subseteq U$.

### 3. Intuitionistic Fuzzy Interior Ideals and Intuitionistic Fuzzy Bi-ideals

**Definition 3.1.** For an IFS $A = (\mu, \gamma)$ in $X$, consider the following axioms:

$$\text{(S1)} \quad \mu(x-y) \geq \min\{\mu(x), \mu(y)\},$$
Then A is called intuitionistic fuzzy subalgebra (IFSA) of X if it satisfies (S1) and (S2) and A is called intuitionistic fuzzy subsemigroup (IFSS) of X if it satisfies (S3) and (S4).

**Theorem 3.2.** Let Y be a non-empty subset of X and $A=(\mu, \gamma)$ be an IFS of X defined by, for all $x \in X$,

$$
\mu(x) = \begin{cases} 
  s, & \text{if } x \in Y \\
  t, & \text{if } x \notin Y 
\end{cases}, \quad \gamma(x) = \begin{cases} 
  s, & \text{if } x \notin Y \\
  t, & \text{if } x \in Y 
\end{cases}
$$

(3.1)

where $s, t \in [0,1]$, $0 \leq s + t \leq 1$ and $s > t$. Then

(i) If Y is a subalgebra of X if and only if $A$ is an IFSA of X,

(ii) If Y is a subsemigroup of X if and only if $A$ is an IFSS of X.

**Proof.** (i) Let Y be a subalgebra of X and $x, y \in X$. If $x, y \in Y$, then since $x - y \in Y$, we have $\mu(x-y) = s = \min \{\mu(x), \mu(y)\}$ and $\gamma(x-y) = t = \max \{\gamma(x), \gamma(y)\}$. If at least one of x and y does not belong to Y, then by the definition of $\mu$ and $\gamma$, we have $\mu(x-y) \geq t = \min \{\mu(x), \mu(y)\}$ and $\gamma(x-y) \leq s = \max \{\gamma(x), \gamma(y)\}$. So (S1) and (S2) are satisfied. Conversely, let $x, y \in Y$. Since A satisfies (S2) and $\gamma(x-y) \leq \max \{\gamma(x), \gamma(y)\} = t$, we have $\gamma(x-y) = t$ by the definition of $\gamma$. Hence we have $x - y \in Y$.

(ii) Let Y be a subsemigroup of X and $x, y \in X$. If $x, y \in Y$, then since $xy \in Y$, we have $\mu(xy) = s = \min \{\mu(x), \mu(y)\}$ and $\gamma(xy) = t = \max \{\gamma(x), \gamma(y)\}$. If at least one of x and y does not belong to Y, then by the definition of $\mu$ and $\gamma$, we have $\mu(xy) \geq t = \min \{\mu(x), \mu(y)\}$ and $\gamma(xy) \leq s = \max \{\gamma(x), \gamma(y)\}$. So (S3) and (S4) are satisfied. Conversely, let $x, y \in Y$. Since A satisfies (S3) and $\mu(xy) \geq \min \{\mu(x), \mu(y)\} = s$, we have $\mu(xy) = s$ by the definition of $\mu$. Hence we have $xy \in Y$.

**Definition 3.3.** An IFS $A=(\mu, \gamma)$ in X is called an intuitionistic fuzzy ideal (IFI) of X if it satisfies (S1), (S2) and the following conditions:

(S5) $\mu(xy) \geq \mu(y)$,
(S6) $\gamma(xy) \leq \gamma(y)$,
(S7) $\mu(xy) \geq \mu(x)$,
(S8) $\gamma(xy) \leq \gamma(x)$

for all $x, y \in X$.

Note that A is an intuitionistic fuzzy left ideal (IFLI) of X if it satisfies (S1), (S2), (S5), (S6) and A is an intuitionistic fuzzy right ideal (IFRI) of X if it satisfies (S1), (S2), (S7), (S8).

**Theorem 3.4.** Let $A=(\mu, \gamma)$ be an IFLI of X. Then the set
is a left ideal of \( X \).

**Proof.** Let \( x, y \in X_\lambda \). Then since \( \mu(x-y) \geq \min \{\mu(x), \mu(y)\} = \mu(0) \) and \( \mu(0)=\mu(0(x-y)) \geq \mu(x-y) \), we get \( \mu(x-y) = \mu(0) \). Also, since \( \gamma(x-y) \leq \max \{\gamma(x), \gamma(y)\} = \gamma(0) \) and \( \gamma(0)=\gamma(0(x-y)) \leq \gamma(x-y) \), we have \( \gamma(x-y) = \gamma(0) \). Hence \( x-y \in X_\lambda \).

For all \( a \in X \) and \( x \in X_\lambda \), since \( \mu(ax) \geq \mu(x) = \mu(0) \), \( \mu(0)=\mu(0(ax)) \geq \mu(ax) \), \( \gamma(ax) \leq \gamma(x) = \gamma(0) \) and \( \gamma(0)=\gamma(0(ax)) \leq \gamma(ax) \), we have \( \mu(ax) = \mu(0) \) and \( \gamma(ax) = \gamma(0) \). Hence \( ax \in X_\lambda \). So \( X_\lambda \) is a left ideal of \( X \).

Similarly, the Theorem 3.4 can be proved for the right case.

**Theorem 3.5.** Let \( Y \) be a non-empty subset of \( X \) and \( A=(\mu, \gamma) \) be an IFS of \( X \) defined by (3.1). Then \( A \) is an IFI of \( X \) if and only if \( Y \) is an ideal of \( X \).

**Proof.** Suppose \( A \) is an IFI of \( X \). By Theorem 3.2 (i), \( Y \) is a subalgebra of \( X \). For all \( y \in Y \) and \( x \in X \), since \( \gamma(xy) \leq \gamma(y) = t \), we have \( \gamma(xy) = t \) by the definition of \( \gamma \). Hence we have \( xy \in Y \). Similarly, it is obtained that \( yx \in Y \) for all \( y \in Y \) and \( x \in X \). So \( Y \) is an ideal of \( X \). Conversely, let \( Y \) be an ideal of \( X \) and \( x, y \in X \). By Theorem 3.2, (S1) and (S2) are satisfied. Also, if at least one of \( x \) and \( y \) belong to \( Y \), since \( Y \) is an ideal of \( X \), we have \( xy \in Y \) and so we get \( \mu(xy) = \mu(x) \) (or \( \mu(y) \) ) and \( \gamma(xy) = \gamma(x) \) (or \( \gamma(y) \) ). If both \( x \) and \( y \) does not belong to \( Y \), then we obtain \( \mu(xy) \geq \mu(x) = \mu(y) \) and \( \gamma(xy) \leq \gamma(x) = \gamma(y) \). Hence (S5)-(S8) are satisfied, and \( A \) is an IFI of \( X \).

**Definition 3.6.** An IFS \( A=(\mu, \gamma) \) in \( X \) is called an intuitionistic fuzzy interior ideal (IFII) of \( X \) if it satisfies (S1), (S2) and the following conditions:

\[
\begin{align*}
(II1) & \quad \mu(xay) \geq \mu(a), \\
(II2) & \quad \gamma(xay) \leq \gamma(a)
\end{align*}
\]
for all \( x, a, y \in X \).

**Theorem 3.7.** Every IFI of \( X \) is an IFII.

**Proof.** Let \( A=(\mu, \gamma) \) be an IFI of \( X \). Then \( A \) satisfies (S1), (S2) and (S5)-(S8). In (S5) and (S6), if we write \( ay \) instead of \( y \) and use (S7) and (S8), we have \( \mu(xay) \geq \mu(ay) \geq \mu(a) \) and \( \gamma(xay) \leq \gamma(ay) \leq \gamma(a) \).

**Theorem 3.8.** If \( X \) is a regular subtraction semigroup, then every IFII of \( X \) is an IFI.

**Proof.** It is clear by (S1), (S2) and Theorem 3.10 in [8].

**Theorem 3.9.** Let \( Y \) be a non-empty subset of \( X \) and \( A=(\mu, \gamma) \) be an IFS of \( X \) defined by (3.1). Then \( Y \) is an interior ideal of \( X \) if and only if \( A=(\mu, \gamma) \) is an IFII of \( X \).

**Proof.** Let \( x, a, y \in X \). If \( a \in Y \), we have \( xay \in Y \) since \( Y \) is an interior ideal. So we get \( \mu(xay) = \mu(a) \) and \( \gamma(xay) = \gamma(a) \). If \( a \notin Y \), then, by the definition of \( \mu \) and \( \gamma \),
\[ \mu(xay) \geq s = \mu(a) \text{ and } \gamma(xay) \leq \gamma(x) \]. Since \( Y \) is also a subalgebra of \( X \), by Theorem 3.2 (i), \( A \) satisfies \((S1)\) and \((S2)\). Conversely, let \( A=(\mu, \gamma) \) be an IFII of \( X \). By Theorem 3.2 (i), \( Y \) is a subalgebra of \( X \). Now let \( xay \) be any element of \( XYX \). Then since \( x \in Y \) and \( A \) is an IFII of \( X \), we have \( \mu(xay) \geq \mu(a) = s \) and hence we get \( \mu(xay) = s \). So it is obtained that \( xay \in Y \).

**Definition 3.10.** An IFS \( A=(\mu, \gamma) \) in \( X \) is called an intuitionistic fuzzy bi-ideal (IFBI) of \( X \) if it satisfies \((S1)\), \((S2)\) and the following conditions:

\[
\begin{align*}
\text{(BI1)} & \quad \mu(xwy) \geq \min \{ \mu(x), \mu(y) \}, \\
\text{(BI2)} & \quad \gamma(xwy) \leq \max \{ \gamma(x), \gamma(y) \}
\end{align*}
\]

for all \( x, w, y \in X \).

**Theorem 3.11.** Let \( Y \) be a non-empty subset of \( X \) and \( A=(\mu, \gamma) \) be an IFS of \( X \) defined by (3.1). Then \( Y \) is a bi-ideal of \( X \) if and only if \( A=(\mu, \gamma) \) is an IFBI of \( X \).

**Proof.** Let \( x, w, y \in X \). If \( x, y \in Y \), we have \( xwy \in Y \) since \( Y \) is a bi-ideal. So we get \( \mu(xwy) = s = \mu(x) = \mu(y) = \min \{ \mu(x), \mu(y) \} \text{ and } \gamma(xwy) = t = \gamma(x) = \gamma(y) = \max \{ \gamma(x), \gamma(y) \} \). If \( x \notin Y \) or \( y \notin Y \), then by the definition of \( \mu \) and \( \gamma \), we obtain \( \mu(xwy) \geq t = \mu(x) = \min \{ \mu(x), \mu(y) \} \) (or \( \mu(xwy) \leq s = \mu(y) = \min \{ \mu(x), \mu(y) \} \)) and \( \gamma(xwy) \leq s = \gamma(x) = \max \{ \gamma(x), \gamma(y) \} \) (or \( \gamma(xwy) \geq t = \gamma(y) = \max \{ \gamma(x), \gamma(y) \} \)). Since \( Y \) is also subalgebra of \( X \), by Theorem 3.2 (i), \( A \) satisfies \((S1)\) and \((S2)\). Conversely, let \( A=(\mu, \gamma) \) be an IFBI of \( X \). By Theorem 3.2 (i), \( Y \) is a subalgebra. Now let \( xwy \) be any element of \( XYX \). Then since \( x, y \in Y \) and \( A \) is an IFBI of \( X \), we have \( \mu(xwy) \geq \min \{ \mu(x), \mu(y) \} = s \) and hence we get \( \mu(xwy) = s \). So it is obtained that \( xwy \in Y \).

Let \( Y \) be a non-empty subset of \( X \) and \( \chi \) be the characteristic function of \( Y \). Then \( A=(\chi, \chi') \) where \( \chi' = 1 - \chi \) is special case of \( A \) defined by (3.1). Therefore \( A=(\chi, \chi') \) satisfies all proved Theorems about \( A \) defined by (3.1).

**References**


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