On Orlicz Difference Sequence Spaces

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Abstract: The main aim of this article is to generalize the famous Orlicz sequence space by using difference operators and a sequence of non-zero scalars and investigate some topological structure relevant to this generalized space.

Key words: Difference sequence space, multiplier sequence space, Orlicz function, AK-BK space, topological isomorphism and Köthe-Toeplitz dual.

Orlicz Fark Dizi Uzayları Üzerine

Özet: Bu makalenin amacı, sıfırdan farklı skalerlerden oluşan bir diziye ve fark operatörlerini kullanarak Orlicz dizi uzaylarını genelleştirmek ve bu yeni tanımladığımız uzayın topolojik yapısını incelemektir.

Anahtar kelimeler: Fark dizi uzayı, çok indisli dizi uzayı, Orlicz fonksiyonu, AK-BK uzayı, topolojik izomorfizm, Köthe-Toeplitz duali.

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1. Introduction

Throughout this paper $w, \ell_\infty, \ell_1, c$ and $c_0$ denote the spaces of all, bounded, absolutely summable, convergent and null sequences $x = (x_k)$ with complex terms respectively. The notion of difference sequence space was introduced by Kizmaz [1], who studied the difference sequence spaces $\ell^\Delta, c^\Delta$ and $c_0^\Delta$, where

$$Z(\Delta) = \{x = (x_k) \in w: (\Delta x_k) \in Z\},$$

where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k$, for $Z=\ell_\infty, c$ and $c_0$.

An Orlicz function $M : [0, \infty) \rightarrow [0, \infty)$ is a function, which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$, for $x > 0$ and $M(x) \rightarrow \infty$, as $x \rightarrow \infty$.

An Orlicz function $M$ can always be represented in the following integral form:

$$M(x) = \int_0^x p(t) \, dt,$$

where $p$, known as kernel of $M$, is right differentiable for $t \geq 0$, $p(0) = 0$, $p(t) > 0$ for $t > 0$, $p$ is non-decreasing, and $p(t) \rightarrow \infty$ as $t \rightarrow \infty$. 
Consider the kernel $p(t)$ associated with the Orlicz function $M(t)$, and let

$$q(s) = \sup \{ t: p(t) \leq s \}$$

Then $q$ possesses the same properties as the function $p$. Suppose now

$$\Phi(x) = \int_0^x q(s) \, ds$$

Then $\Phi$ is an Orlicz function. The functions $M$ and $\Phi$ are called mutually complementary Orlicz functions.

Now we state the following well known results which can be found in [2]. Let $M$ and $\Phi$ are mutually complementary Orlicz functions. Then we have (Young’s inequality)

(i) For $x, y \geq 0$, $xy \leq M(x) + \Phi(y)$

We also have

(ii) For $x \geq 0$, $xp(x) = M(x) + \Phi(p(x))$

(iii) $M(\lambda x) < \lambda M(x)$

for all $x \geq 0$ and $\lambda$ with $0 < \lambda < 1$.

An Orlicz function $M$ is said to satisfy the $\Delta_2$-condition for small $x$ or at 0 if for each $k > 0$ there exist $R_k > 0$ and $x_k > 0$ such that

$$M(kx) \leq R_k M(x)$$

for all $x \in (0, x_k]$.

Moreover an Orlicz function $M$ is said to satisfy the $\Delta_2$-condition if and only if

$$\limsup_{x \to 0} \frac{M(2x)}{M(x)} < \infty.$$ 

Two Orlicz functions $M_1$ and $M_2$ are said to be equivalent if there are positive constants $\alpha, \beta$ and $x_0$ such that

$$M_1(\alpha x) \leq M_2(x) \leq M_1(\beta x)$$

for all $x$ with $0 \leq x \leq x_0$.

Lindenstrauss and Tzafriri [3] used the Orlicz function and introduced the sequence space $\ell_{M}$ as follows:

$$\ell_{M} = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}.$$ 

For more details about Orlicz functions and sequence spaces associated with Orlicz functions one may refer to [2-5].

Let $\Lambda = (\lambda_k)$ be a sequence of non-zero scalars. Then for a sequence space $E$, the multiplier sequence space $E(\Lambda)$, associated with the multiplier sequence $\Lambda$ is defined as

$$E(\Lambda) = \left\{ (x_k) \in w : (\lambda_k x_k) \in E \right\}.$$
The scope for the studies on sequence spaces was extended by using the notion of associated multiplier sequences. Goes and Goes [6] defined the differentiated sequence space \( dE \) and integrated sequence space \( \int E \) for a given sequence space \( E \), using the multiplier sequences \((k^{-1})\) and \((k)\) respectively. A multiplier sequence can be used to accelerate the convergence of the sequences in some spaces. In some sense, it can be viewed as a catalyst, which is used to accelerate the process of chemical reaction. Sometimes the associated multiplier sequence delays the rate of convergence of a sequence. Thus it also covers a larger class of sequences for study. In the present article we shall consider a general multiplier sequence \( \Lambda = (\lambda_k) \) of non-zero scalars.

The notion of duals of sequence spaces was introduced by Köthe and Toeplitz [7]. Later on it was studied by Kizmaz [1], Kamthan [8] and many others.

Let \( E \) and \( F \) be two sequence spaces. Then the \( F \) dual of \( E \) is defined as
\[
E^F = \{ (x_k) \in w : (x_k y_k) \in F \text{ for all } (y_k) \in E \}.
\]
For \( F = \ell_1 \), the dual is termed as Köthe-Toeplitz or \( \alpha \)-dual of \( E \) and denoted by \( E^\alpha \). More precisely, we have the following definition of Köthe-Toeplitz dual of \( E \):
\[
E^\alpha = \left\{ a = (a_k) : \sum \lambda_k x_k < \infty, \text{ for all } x \in E \right\}.
\]
It is known that if \( X \subseteq Y \), then \( Y^\alpha \subseteq X^\alpha \). If \( E^{FF} = E \), where \( E^{FF} = (E^F)^F \), then \( E \) is said to be \( F \)-reflexive or \( F \)-perfect. In particular, if \( E^{\alpha\alpha} = E \), then \( E \) is also said to be a Köthe space.

Let \( \Lambda = (\lambda_k) \) be a sequence of non-zero scalars. Then we define the following spaces.

**Definition 1.1.** Let \( M \) be any Orlicz function. Then we define
\[
\ell_M (\Delta, \Lambda) = \left\{ x \in w : \delta^\Lambda (M, x) = \sum_{k=1}^\infty M (|\Delta \lambda_k x_k|) < \infty \right\},
\]
where \( \Delta \lambda_k x_k = \lambda_k x_k - \lambda_{k+1} x_{k+1} \) for all \( k \geq 1 \).

We can write \( \ell_M (\Delta^0, \Lambda) = \ell_M (\Lambda) \) and if \( \lambda_k = 1 \) for all \( k \geq 1 \), then we write \( \ell_M (\Delta^0, \Lambda) = \ell_M \).

Similarly we can define \( \ell_M (\nabla, \Lambda) \), where \( \nabla \lambda_k x_k = \lambda_k x_k - \lambda_{k-1} x_{k-1} \) for all \( k \geq 1 \).

**Definition 1.2.** Let \( M \) and \( \Phi \) be mutually complementary functions. Then we define
\[
\ell_M (\Delta, \Lambda) = \left\{ x \in w : \sum_{k=1}^\infty (\Delta \lambda_k x_k) y_k \text{ converges for all } y \in \ell_\Phi \right\}.
\]
We call this sequence space as Orlicz difference sequence space associated with the multiplier sequence \( \Lambda = (\lambda_k) \).

We can write \( \ell_M (\Delta^0, \Lambda) = \ell_M (\Lambda) \) and if \( \lambda_k = 1 \) for all \( k \geq 1 \), then we write
\[ \ell_M \left( \Delta^0, \Lambda \right) = \ell_M. \]

Similarly we can define \( \ell_M \left( \nabla, \Lambda \right) \) where \( \nabla \lambda_i x_k = \lambda_i x_k - \lambda_{i-1} x_{k-1} \) for all \( k \geq 1 \).

One can easily observe in the special case \( M(x) = x^p \) with \( 0 < p < \infty \) and \( \Lambda = (\lambda_i) = (1,1,\ldots) = e \), the sequence space \( \ell_M \left( \nabla, \Lambda \right) \) is reduced in the case \( 1 \leq p < \infty \) to the Banach space \( bv_p \) introduced by Başar and Altay [9] and is reduced in the case \( 0 < p < 1 \) to the \( p \)-normed complete space \( bv_p \) introduced by Altay and Başar [10], where \( bv_p \) denotes the space of all sequences \( x = (x_k) \) such that
\[ \nabla x = (x_k - x_{k-1}) \in \ell_p. \]

2. Main Results

In this section we investigate the main results of this article.

Proposition 2.1. For any Orlicz function \( M \),

(i) \( \ell_M \left( \Delta, \Lambda \right) \subset \ell_M \left( \Delta, \Lambda \right) \),

(ii) \( \ell_M \left( \nabla, \Lambda \right) \subset \ell_M \left( \nabla, \Lambda \right) \).

Proof: (i) Let \( x \in \ell_M \left( \Delta, \Lambda \right) \). Then \( \sum_{k=1}^{\infty} M \left( |\Delta \lambda_i x_k| \right) < \infty \). Now using (1), we have
\[ \left| \sum_{k=1}^{\infty} (\Delta \lambda_i x_k) y_k \right| \leq \sum_{k=1}^{\infty} |(\Delta \lambda_i x_k) y_k| \leq \sum_{k=1}^{\infty} M \left( |\Delta \lambda_i x_k| \right) + \sum_{k=1}^{\infty} \Phi \left( |y_k| \right) < \infty, \]
for every \( y = (y_k) \) with \( y \in \ell_\Phi \). Thus \( x \in \ell_M \left( \Delta, \Lambda \right) \).

(ii) Since the proof is similar to the proof of part (i), we omit it.

Proposition 2.2. (i) For each \( x \in \ell_M \left( \Delta, \Lambda \right) \), sup \( \left\{ \left| \sum_{i=1}^{\infty} (\Delta \lambda_i x_i) y \right| : \delta \left( \Phi, y \right) \leq 1 \right\} < \infty, \)

(ii) For each \( x \in \ell_M \left( \nabla, \Lambda \right) \), sup \( \left\{ \left| \sum_{i=1}^{\infty} (\nabla \lambda_i x_i) y_i \right| : \delta \left( \Phi, y \right) \leq 1 \right\} < \infty. \)

Proof: (i) Suppose that the result is not true. Then for each \( n \geq 1 \), there exists \( y^n \) with \( \delta \left( \Phi, y^n \right) \leq 1 \) such that
\[ \left| \sum_{i=1}^{\infty} (\Delta \lambda_i x_i) y^n_i \right| > 2^n. \]
Without loss of generality we may assume that \( (\Delta \lambda_i x_i), y^n \geq 0 \). Now, we can define a sequence \( z = \{z_i\} \) by
\[ z_i = \sum_{n=1}^{\infty} x_i \frac{1}{2^n} y^n_i. \]
By the convexity of $\Phi$,

$$\Phi\left(\sum_{n=1}^{l} \frac{1}{2^n} y^n_i\right) \leq \frac{1}{2} \left[\Phi(y^1_i) + \Phi\left(\frac{y^2_i}{2} + \ldots + \frac{y^l_i}{2^{l-1}}\right)\right] \leq \sum_{n=1}^{l} \frac{1}{2^n} \Phi(y^n_i)$$

and hence, using the continuity of $\Phi$, we have

$$\delta(\Phi, z) = \sum_{i=1}^{\infty} \Phi(z_i) \leq \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{2^n} \Phi(y^n_i) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$  

But for every $l \geq 1$,

$$\sum_{i=1}^{\infty} (\Delta \lambda x_i) z_i \geq \sum_{i=1}^{\infty} (\Delta \lambda x_i) \sum_{n=1}^{l} \frac{1}{2^n} y^n_i = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} (\Delta \lambda x_i) \frac{y^n_i}{2^n} \geq l.$$

Hence $\sum_{i=1}^{\infty} (\Delta \lambda x_i) z_i$ diverges and this implies that $x \notin \ell_{M} (\Delta, \Lambda)$. This contradiction leads us to the required result.

(ii) Proof is similar to that of part (i).

The preceding result encourage us to introduce the following norms $\| \|_{\Lambda}^{\Delta}$ and $\| \|_{\Lambda}^{\nabla}$ on $\ell_{M} (\Delta, \Lambda)$ and $\ell_{M} (\nabla, \Lambda)$, respectively.

**Proposition 2.3.**

(i) $\ell_{M} (\Delta, \Lambda)$ is a normed linear space under the norm $\| \|_{M}^{\Delta}$ defined by

$$\| x \|_{M}^{\Delta} = |\lambda x_i| + \sup \left\{ \sum_{i=1}^{\infty} (\Delta x_i) y_i : \delta(\Phi, y) \leq 1 \right\}.$$  

(ii) $\ell_{M} (\nabla, \Lambda)$ is a normed linear space under the norm $\| \|_{M}^{\nabla}$ defined by

$$\| x \|_{M}^{\nabla} = \sup \left\{ \sum_{i=1}^{\infty} (\nabla lambda x_i) y_i : \delta(\Phi, y) \leq 1 \right\}.$$  

**Proof.**

(i) It is easy to verify that $\ell_{M} (\Delta, \Lambda)$ is a linear space. Now we show that $\| \|_{M}^{\Delta}$ is a norm on $\ell_{M} (\Delta, \Lambda)$.

If $x = \theta$, then obviously $\| x \|_{M}^{\Delta} = 0$. Conversely assume $\| x \|_{M}^{\Delta} = 0$. Then using the definition of norm, we have

$$|\lambda x_i| + \sup \left\{ \sum_{i=1}^{\infty} (\Delta x_i) y_i : \delta(\Phi, y) \leq 1 \right\} = 0.$$

This implies

$$|\lambda x_i| = 0$$

(7)

and

$$\sup \left\{ \sum_{i=1}^{\infty} (\Delta x_i) y_i : \delta(\Phi, y) \leq 1 \right\} = 0.$$
This implies that \( \sum_{i=1}^{\infty} (\Delta \lambda_i x_i) y_i = 0 \) for all \( y \) such that \( \delta(\Phi, y) \leq 1 \).

Now considering \( y = \{e_i\} \) if \( \Phi(1) \leq 1 \) otherwise considering \( y = \{e_i/\Phi(1)\} \) so that
\[
\Delta \lambda_i x_i = 0 \text{ for all } i \geq 1. \tag{8}
\]

Combining (7) and (8), we have \( x_i = 0 \) for all \( i \geq 1 \), since \( (\lambda_k) \) is a sequence of non-zero scalars and thus \( x = 0 \).

It is easy to show
\[
\|\alpha x\|_M^\Lambda = |\alpha| \|x\|_M^\Lambda \text{ and } \|x + y\|_M^\Lambda \leq \|x\|_M^\Lambda + \|y\|_M^\Lambda.
\]

(ii) Let \( x = \theta \), then obviously \( \|x\|_M^\Lambda = 0 \). Conversely assume \( \|x\|_M^\Lambda = 0 \). Then using the definition of norm, we have
\[
\sup \left\{ \sum_{i=1}^{\infty} (\nabla \lambda_i x_i) y_i : \delta(\Phi, y) \leq 1 \right\} = 0.
\]
This implies \( \sum_{i=1}^{\infty} (\nabla \lambda_i x_i) y_i = 0 \) for all \( y \) such that \( \delta(\Phi, y) \leq 1 \).

Now considering \( y = \{e_i\} \) if \( \Phi(1) \leq 1 \) otherwise considering \( y = \{e_i/\Phi(1)\} \) so that
\[
\nabla \lambda_i x_i = 0 \text{ for all } i \geq 1.
\]
Taking \( i = 1 \), we have
\[
\nabla \lambda_1 x_1 = \lambda_1 x_1 - \lambda_0 x_0 = 0.
\]
This implies \( \lambda_1 x_1 = 0 \), by taking \( x_0 = 0 \). Proceeding in this way we have \( \lambda_i x_i = 0 \) for all \( i \geq 1 \) and so \( x_i = 0 \) for all \( i \geq 1 \), since \( (\lambda_k) \) is a sequence of non-zero scalars. Thus \( x = \theta \).

It is easy to show
\[
\|\alpha x\|_M^\Lambda = |\alpha| \|x\|_M^\Lambda \text{ and } \|x + y\|_M^\Lambda \leq \|x\|_M^\Lambda + \|y\|_M^\Lambda.
\]
This completes the proof.

**Remark.** \( \sum_{k=1}^{\infty} (\Delta \lambda_k x_k) y_k < \infty \) for all \( y \in \ell_\Phi \) if and only if \( \sum_{k=1}^{\infty} (\nabla \lambda_k x_k) y_k < \infty \) for all \( y \in \ell_\Phi \).

Also it is obvious that the norms \( \|\cdot\|_M^\Lambda \) and \( \|\cdot\|_M^\Lambda \) are equivalent.

**Proposition 2.4.** (i) \( \ell_M (\Delta, \Lambda) \) is a Banach space under the norm \( \|\cdot\|_M^\Lambda \),
(ii) \( \ell_M (\nabla, \Lambda) \) is a Banach space under the norm \( \|\cdot\|_M^\Lambda \).

**Proof.** We shall give proof of part (i). Proof of part (ii) is easy than part (i).

Let \( (x^j) \) be any Cauchy sequence in \( \ell_M (\Delta, \Lambda) \). Then for any \( \varepsilon > 0 \), there exists a positive integer \( n_0 \) such that
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\[ \|x' - x'^\prime\| \leq \epsilon, \]

for all \( i, j \geq n_0 \). Using the definition of norm, we get

\[ |\tilde{\lambda}_i(x'_i - x'^\prime_i)| + \sup \left\{ \sum_{k=1}^{\infty} (\Delta \tilde{\lambda}_k(x'_k - x'^\prime_k))y_k \right\} : \delta(\Phi, y) \leq 1 \leq 2\epsilon, \]

for all \( i, j \geq n_0 \). This implies that \(|\tilde{\lambda}_i(x'_i - x'^\prime_i)| < \epsilon, \) for all \( i, j \geq n_0 \). Thus \( \lambda_i x'_i \) is a Cauchy sequence in \( C \) and hence it is a convergent sequence in \( C \).

Let

\[ \lim_{i \to \infty} \tilde{\lambda}_i x'_i = z_i. \] (9)

Again we have

\[ \sup \left\{ \sum_{k=1}^{\infty} (\Delta \tilde{\lambda}_k(x'_k - x'^\prime_k))y_k \right\} : \delta(\Phi, y) \leq 1 \leq \epsilon \]

for all \( i, j \geq n_0 \) and so

\[ \sum_{k=1}^{\infty} (\Delta \tilde{\lambda}_k(x'_k - x'^\prime_k))y_k < \epsilon \]

for all \( y \) with \( \delta(\Phi, y) \leq 1 \) and \( i, j \geq n_0 \).

Now considering \( y = \{e_i\} \) if \( \Phi(1) \leq 1 \) otherwise considering \( y = \{e_i/\Phi(1)\} \) we have \( \Delta \tilde{\lambda}_k x'_k \) is a Cauchy sequence in \( C \) for all \( k \geq 1 \) and hence it is a convergent sequence in \( C \) for all \( k \geq 1 \).

Let

\[ \lim_{i \to \infty} \Delta \tilde{\lambda}_k x'_k = y_k \] (10)

for all \( k \geq 1 \). Using (9) and (10) we have \( \lim_{i \to \infty} \tilde{\lambda}_i x'_i \) exists for each \( k \geq 1 \) and so \( \lim_{i \to \infty} x'_i = x_k \), say exists for each \( k \geq 1 \).

Now

\[ \lim_{j \to \infty} |\tilde{\lambda}_i(x'_i - x'_i)| = |\tilde{\lambda}_i(x'_i - x_i)| < \epsilon \]

for all \( i \geq n_0 \). Also we can have

\[ \sup \left\{ \sum_{k=1}^{\infty} (\Delta \tilde{\lambda}_k(x'_k - x_k))y_k \right\} : \delta(\Phi, y) \leq 1 \leq 2\epsilon \]

for all \( i \geq n_0 \) as \( j \to \infty \). Thus

\[ |\tilde{\lambda}_i(x'_i - x_i)| + \sup \left\{ \sum_{k=1}^{\infty} (\Delta \tilde{\lambda}_k(x'_k - x_k))y_k \right\} : \delta(\Phi, y) \leq 1 \leq 2\epsilon \]

for all \( i \geq n_0 \) and as \( j \to \infty \). It follows that \((x^l - x) \in \ell_M(\Delta, \Lambda)\) and \( \ell_M(\Delta, \Lambda) \) is a linear space and hence \( x = (x_k) \in \ell_M(\Delta, \Lambda) \).
From above proof we can easily conclude that \( \|x\|^3_{M_1} \to 0 \) implies that \( x_i' \to 0 \) for each \( i \geq 1 \). Hence we have the following Proposition.

**Proposition 2.5.** \( \ell_M(\Delta, \Lambda) \) and \( \ell_M(\nabla, \Lambda) \) are BK spaces under the norms defined by (5) and (6), respectively.

Our next aim is to show that \( \ell_M(\Delta, \Lambda) \) and \( \ell_M(\nabla, \Lambda) \) can be made BK spaces under different but equivalent norms.

**Proposition 2.6.**

(i) \( \ell_M(\Delta, \Lambda) \) is a normed linear space under the norm \( \|x\|^3_{M_1} \) defined by

\[
\|x\|^3_{M_1} = |\lambda_i x_i| + \inf \left\{ \rho > 0 : \sum_{k=1}^\infty M \left( \frac{\Delta \lambda_k x_k}{\rho} \right) \leq 1 \right\},
\]

(ii) \( \ell_M(\nabla, \Lambda) \) is a normed linear space under the norm \( \|x\|^\nabla_{M_1} \) defined by

\[
\|x\|^\nabla_{M_1} = \inf \left\{ \rho > 0 : \sum_{k=1}^\infty M \left( \frac{\nabla \lambda_k x_k}{\rho} \right) \leq 1 \right\}.
\]

**Proof.** (i) Clearly \( \|x\|^3_{M_1} = 0 \) if \( x = \theta \). Next suppose \( \|x\|^3_{M_1} = 0 \). Then from (11) we have

\[
|\lambda_i x_i| = 0 \text{ and so } \lambda_i x_i = 0.
\]

Again \( \inf \left\{ \rho > 0 : \sum_{k=1}^\infty M \left( \frac{\Delta \lambda_k x_k}{\rho} \right) \leq 1 \right\} = 0 \). This implies that for a given \( \varepsilon > 0 \), there exists some \( \rho_\varepsilon (0 < \rho_\varepsilon < \varepsilon) \) such that

\[
\sup_k M \left( \frac{\Delta \lambda_k x_k}{\rho_\varepsilon} \right) \leq 1.
\]

This implies that \( M \left( \frac{\Delta \lambda_k x_k}{\rho_\varepsilon} \right) \leq 1 \) for all \( k \geq 1 \). Thus

\[
M \left( \frac{\Delta \lambda_k x_k}{\varepsilon} \right) \leq M \left( \frac{\Delta \lambda_k x_k}{\rho_\varepsilon} \right) \leq 1
\]

for all \( k \geq 1 \).

Suppose \( \Delta \lambda_n x_n \neq 0 \), for some \( i \). Let \( \varepsilon \to 0 \), then \( \frac{\Delta \lambda_n x_n}{\varepsilon} \to \infty \). It follows that

\[
M \left( \frac{\Delta \lambda_n x_n}{\varepsilon} \right) \to \infty \text{ as } \varepsilon \to 0 \text{ for some } n_i \in N. \text{ This is a contradiction. Therefore}
\]

\[
\Delta \lambda_k x_k = 0
\]

(14)
for all $k \geq 1$. Thus, by (13) and (14), it follows that $\lambda_k x_k = 0$ for all $k \geq 1$. Hence $x = \theta$, since $(\lambda_k)$ is a sequence of non-zero scalars.

Let $x = (x_k)$ and $y = (y_k)$ be any two elements of $\ell_M (\Lambda, \Lambda)$. Then there exist $\rho_1, \rho_2 > 0$ such that

$$
\sup_k M \left( \frac{|\Delta \lambda_k x_k|}{\rho_1} \right) \leq 1 \quad \text{and} \quad \sup_k M \left( \frac{|\Delta \lambda_k y_k|}{\rho_2} \right) \leq 1.
$$

Let $\rho = \rho_1 + \rho_2$. Then by convexity of $M$, we have

$$
\sup_k M \left( \frac{|\Delta \lambda_k (x_k + y_k)|}{\rho} \right) \leq \frac{\rho_1}{\rho_1 + \rho_2} \sup_k M \left( \frac{|\Delta \lambda_k x_k|}{\rho_1} \right) + \frac{\rho_2}{\rho_1 + \rho_2} \sup_k M \left( \frac{|\Delta \lambda_k y_k|}{\rho_2} \right) \leq 1.
$$

Hence we have

$$
\|x + y\|_M = |\lambda(x_1 + y_1)| + \inf \left\{ \rho > 0 : \sup_k M \left( \frac{|\Delta \lambda_k (x_k + y_k)|}{\rho} \right) \leq 1 \right\}
$$

$$
\leq |\lambda x_1| + \inf \left\{ \rho_1 > 0 : \sup_k M \left( \frac{|\Delta \lambda_k x_k|}{\rho_1} \right) \leq 1 \right\} + |\lambda y_1|
$$

$$
+ \inf \left\{ \rho_2 > 0 : \sup_k M \left( \frac{|\Delta \lambda_k y_k|}{\rho_2} \right) \leq 1 \right\}.
$$

This implies $\|x + y\|_M \leq \|x\|_M + \|y\|_M$.

Finally, let $\nu$ be any scalar. Then

$$
\|\nu x\|_M = |\nu \lambda x_1| + \inf \left\{ \rho > 0 : \sup_k M \left( \frac{|\Delta \nu \lambda_k x_k|}{\rho} \right) \leq 1 \right\}
$$

$$
= |\nu| |\lambda x_1| + \inf \left\{ r |\nu| > 0 : \sup_k M \left( \frac{|\Delta \lambda_k x_k|}{r} \right) \leq 1 \right\}
$$

$$
= |\nu| \|x\|_M.
$$

where $r = \frac{\rho}{|\nu|}$. This completes the proof.

(ii) Proof is easy than part (i).

**Remark.** It is obvious that the norms $\|x\|_M$ and $\|\nu x\|_M$ are equivalent.

**Proposition 2.7.** For $x \in \ell_M (\nabla, \Lambda)$, we have
Proof. Proof is immediate from (12).

Now we show that the norms \( \|x\|_M^\vee \) and \( \|x\|_M^{\vee} \) are equivalent. To prove this some other results are required. First we prove those results.

**Proposition 2.8.** Let \( x \in \ell_M(\nabla, \Lambda) \) with \( \|x\|_M^\vee \leq 1 \). Then \( \left\{ p\left(\nabla \lambda_n x_n\right)\right\} \in \ell_{\Phi} \) and

\[
\delta\left(\Phi, \left\{ p\left(\nabla \lambda_n x_n\right)\right\}\right) \leq 1.
\]

**Proof.** For any \( z \in \tilde{\ell}_{\Phi} \), we may write

\[
\left|\sum_{i=1}^\infty (\nabla \lambda_i) z_i\right| \leq \left\|x\right\|_M^\vee \delta(\Phi, z) \text{ if } \delta(\Phi, z) \leq 1 \quad \text{and} \quad \left|\sum_{i=1}^\infty (\nabla \lambda_i) z_i\right| \leq \left\|x\right\|_M^\vee \delta(\Phi, z) \text{ if } \delta(\Phi, z) > 1.
\]

(15)

Let now \( x \in \ell_M(\nabla, \Lambda) \) with \( \|x\|_M^\vee \leq 1 \). Also \( x^{(n)} = (x_1, \ldots, x_n, 0, 0, \ldots) \in \ell_M(\nabla, \Lambda) \) for \( n \geq 1 \).

We observe that

\[
\left\|x^{(n)}\right\|_M^\vee \geq \left|\sum_{i=1}^\infty (\nabla \lambda_i) y_i^{(n)}\right| = \left|\sum_{i=1}^\infty (\nabla \lambda_i x_i^{(n)}) y_i\right|, \quad n \geq 1
\]

for every \( y \in \tilde{\ell}_{\Phi} \) with \( \delta(\Phi, y) \leq 1 \) and thus

\[
\left\|x^{(n)}\right\|_M^\vee \leq \left\|x\right\|_M^\vee \leq 1.
\]

Since

\[
\sum_{i=1}^n \Phi\left( p\left(\nabla \lambda_i x_i\right)\right) = \sum_{i=1}^\infty \Phi\left( p\left(\nabla \lambda_i x_i^{(n)}\right)\right).
\]

We find that \( \left\{ p\left(\nabla \lambda_i x_i^{(n)}\right)\right\} \in \ell_{\Phi} \) for each \( n \geq 1 \). Let \( l \geq 1 \) be an integer such that

\[
\sum_{i=1}^l \Phi\left( p\left(\nabla \lambda_i x_i\right)\right) > 1.
\]

Then \( \sum_{i=1}^\infty \Phi\left( p\left(\nabla \lambda_i x_i^{(n)}\right)\right) > 1 \). Using (2), we have

\[
\Phi\left( p\left(\nabla \lambda_i x_i^{(n)}\right)\right) < M \left(\left|\nabla \lambda_i x_i^{(n)}\right|\right) + \Phi\left( p\left(\nabla \lambda_i x_i^{(n)}\right)\right) = \left|\nabla \lambda_i x_i^{(n)}\right| p\left(\nabla \lambda_i x_i^{(n)}\right)
\]

for all \( i, l \geq 1 \). So by (15), we get

\[
\sum_{i=1}^\infty \Phi\left( p\left(\nabla \lambda_i x_i^{(n)}\right)\right) \leq \left\|x^{(n)}\right\|_M^\vee \delta\left(\Phi, \left\{ p\left(\nabla \lambda_i x_i^{(n)}\right)\right\}\right).
\]
This implies that \( \| x^{(i)} \|_M^Y > 1 \), a contradiction. This contradiction implies that

\[
\sum_{i=1}^l \Phi \left( p \left( \| \nabla \varphi_i \| \right) \right) \leq 1
\]

for all \( l \geq 1 \). Hence \( \left\{ p \left( \| \nabla \varphi_i \| \right) \right\} \in \ell_{\Phi} \) and \( \delta \left( \Phi, \left\{ p \left( \| \nabla \varphi_i \| \right) \right\} \right) \leq 1. \)

**Proposition 2.9.** Let \( x \in \ell_M (\nabla, \Lambda) \) with \( \| x \|_M^Y \leq 1. \) Then \( x \in \ell_M (\nabla, \Lambda) \) and \( \delta^Y (M, x) \leq \| x \|_M^Y. \)

**Proof.** Let \( y = \left\{ p \left( \| \nabla \varphi_i \| / \text{sgn} (\nabla \varphi_i) \right) \right\}. \) Then from Proposition 2.8, \( y \in \ell_{\Phi} \) and \( \delta (\Phi, y) \leq 1. \) By (2), we get

\[
\sum_{i=1}^\infty M \left( \| \nabla \varphi_i \| \right) \leq \sum_{i=1}^\infty M \left( \| \nabla \varphi_i \| \right) + \sum_{i=1}^\infty \Phi \left( p \left( \| \nabla \varphi_i \| \right) \right)
\]

\[
= \sum_{i=1}^\infty \left( \| \nabla \varphi_i \| \right) \left( p \left( \| \nabla \varphi_i \| \right) \right)
\]

\[
= \sum_{i=1}^\infty (\nabla \varphi_i) y_i \leq \| x \|_M^Y.
\]

This implies that \( \delta^Y (M, x) \leq \| x \|_M^Y. \)

**Proposition 2.10.** For \( x \in \ell_M (\nabla, \Lambda) \), we have

\[
\sum_{i=1}^\infty M \left( \| \nabla \varphi_i \| x \|_M^Y \right) \leq \| x \|_M^Y \leq 2 \| x \|_M^Y.
\]

**Proof.** Proof is immediate from Proposition 2.9.

**Theorem 2.11.** For \( x \in \ell_M (\nabla, \Lambda) \), \( \| x \|_M^Y \leq \| x \|_M^Y \leq 2 \| x \|_M^Y. \)

**Proof.** We have

\[
\| x \|_M^Y = \inf \left\{ \rho > 0 : \sum_{i=1}^\infty M \left( \frac{\| \nabla \varphi_i \| x \|_M^Y}{\rho} \right) \leq 1 \right\}.
\]

Then using Proposition 2.10, we get

\[
\| x \|_M^Y \leq \| x \|_M^Y.
\]

Let us suppose that \( x \in \ell_M (\nabla, \Lambda) \) with \( \| x \|_M^Y \leq 1. \) Then \( x \in \ell_M (\nabla, \Lambda) \) and \( \delta^Y (M, x) \leq 1. \)

Indeed,

\[
\frac{1}{\| x \|_M^Y} \sum_{i=1}^\infty M \left( \| \nabla \varphi_i \| \right) \leq \sum_{i=1}^\infty M \left( \frac{\| \nabla \varphi_i \| x \|_M^Y}{\| x \|_M^Y} \right) \leq 1,
\]

by Proposition 2.7.
Thus $\frac{x}{\|x\|_{(M)}} \in \tilde{\ell}_M(\nabla, \Lambda)$ with $\delta \left( M, \frac{x}{\|x\|_{(M)}} \right) \leq 1$. We further observe that for an arbitrary $z \in \tilde{\ell}_M(\nabla, \Lambda)$,

$$\|z\|_M^\vee = \sup \left\{ \sum_{i=1}^\infty (\nabla \lambda_i z_i) y_i : \delta(\Phi, y) \leq 1 \right\} \leq 1 + \delta_Y^\vee (M, z)$$

using (1). Hence taking $z = \frac{x}{\|x\|_{(M)}}$, we have

$$\left\| \frac{x}{\|x\|_{(M)}} \right\|_{(M)}^\vee \leq 1 + \sum_{i=1}^\infty M \left( \frac{|x|}{\|x\|_{(M)}} \right) \leq 2$$

by Proposition 2.7. Thus $\|x\|_{(M)}^\vee \leq 2 \|x\|_{(M)}^\vee$. This completes the proof.

**Proposition 2.12.** For any Orlicz function $M$, $\ell_M(\nabla, \Lambda) = \ell_M'(\nabla, \Lambda)$, where

$$\ell_M'(\nabla, \Lambda) = \left\{ x \in \mathbb{R} : \sum_{k=1}^\infty M \left( \frac{[\nabla \lambda_k x_k]}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}.$$

**Proof.** Proof follows from Proposition 2.10.

In view of above Proposition we give the following definition.

**Definition 2.13.** For any Orlicz function $M$,

$$h_M(\nabla, \Lambda) = \left\{ x \in \mathbb{R} : \sum_{k=1}^\infty M \left( \frac{[\nabla \lambda_k x_k]}{\rho} \right) < \infty, \text{ for each } \rho > 0 \right\}.$$

Clearly $h_M(\nabla, \Lambda)$ is a subspace of $\ell_M(\nabla, \Lambda)$. Henceforth we shall write $\| \cdot \|$ instead of $\| \cdot \|_{(M)}$ provided it does not lead to any confusion. The topology of $h_M(\nabla, \Lambda)$ is the one it inherits from $\| \cdot \|$.

**Proposition 2.14.** Let $M$ be an Orlicz function. Then $(h_M(\nabla, \Lambda), \| \cdot \|)$ is an $AK$-$BK$ space.

**Proof.** First we show that $h_M(\nabla, \Lambda)$ is an $AK$ space. Let $x \in h_M(\nabla, \Lambda)$. Then for each $\varepsilon$, $0 < \varepsilon < 1$, we can find an $n_0$ such that

$$\sum_{i=1}^{\infty} M \left( \frac{|\nabla \lambda_i x_i|}{\varepsilon} \right) \leq 1.$$

Hence for $n \geq n_0$,

$$\|x-x^{(n)}\| = \inf \left\{ \rho > 0 : \sum_{i \geq n+1} M \left( \frac{|\nabla \lambda_i x_i|}{\rho} \right) \leq 1 \right\} \leq \inf \left\{ \rho > 0 : \sum_{i \geq n} M \left( \frac{|\nabla \lambda_i x_i|}{\rho} \right) \leq 1 \right\} < \varepsilon.$$
Thus we can conclude that $h_M(\nabla, \Lambda)$ is an AK space. Next to show $h_M(\nabla, \Lambda)$ is a BK space it is enough to show $h_M(\nabla, \Lambda)$ is a closed subspace of $h_M(\nabla, \Lambda)$. For this let $\{x^n\}$ be a sequence in $h_M(\nabla, \Lambda)$ such that $\|x^n-x\| \to 0$, where $x \in h_M(\nabla, \Lambda)$. To complete the proof we need to show that $x \in h_M(\nabla, \Lambda)$, i.e.,

$$\sum_{i\geq 1} M\left(\frac{\nabla \lambda, x_i}{\rho}\right) < \infty$$

for every $\rho > 0$. To $\rho > 0$ there corresponds an $l$ such that $\|x'-x\| \leq \frac{\rho}{2}$. Then using convexity of $M$,

$$\sum_{i\geq 1} M\left(\frac{\nabla \lambda, x_i}{\rho}\right) = \sum_{i\geq 1} M\left(\frac{2|\nabla \lambda, x_i'|-2(\nabla \lambda, x_i'-\nabla \lambda, x_i)|}{2\rho}\right)$$

$$\leq \frac{1}{2} \sum_{i\geq 1} M\left(\frac{2|\nabla \lambda, x_i'|}{\rho}\right) + \frac{1}{2} \sum_{i\geq 1} M\left(\frac{2|\nabla \lambda, (x_i'-x_i)|}{\rho}\right)$$

$$\leq \frac{1}{2} \sum_{i\geq 1} M\left(\frac{2|\nabla \lambda, x_i'|}{\rho}\right) + \frac{1}{2} \sum_{i\geq 1} M\left(\frac{2|\nabla \lambda, (x_i'-x)|}{\rho}\right) < \infty$$

by proposition 2.7. Thus $x \in h_M(\nabla, \Lambda)$ and consequently $h_M(\nabla, \Lambda)$ is a BK space.

**Proposition 2.15.** Let $M$ be an Orlicz function. If $M$ satisfies the $\Delta_2$-condition at 0, then $\ell_M(\nabla, \Lambda)$ is an AK space.

**Proof.** In fact we shall show that if $M$ satisfies the $\Delta_2$-condition at 0, then $\ell_M(\nabla, \Lambda) = h_M(\nabla, \Lambda)$ and the result follows. Therefore it is enough to show that $\ell_M(\nabla, \Lambda) \subset h_M(\nabla, \Lambda)$. Let $x \in \ell_M(\nabla, \Lambda)$, then $\rho > 0$,

$$\sum_{i\geq 1} M\left(\frac{\nabla \lambda, x_i}{\rho}\right) < \infty.$$ 

This implies that

$$M\left(\frac{\nabla \lambda, x_i}{\rho}\right) \to 0 \text{ as } i \to \infty. \quad (16)$$

Choose an arbitrary $l > 0$. If $\rho \leq l$, then $\sum_{i\geq 1} M\left(\frac{\nabla \lambda, x_i}{\rho l}\right) < \infty$. Let now $l < \rho$ and put $k = \frac{\rho}{l}$.

Since $M$ satisfies $\Delta_2$-condition at 0, there exist $R = R_k > 0$ and $r = r_k > 0$ with $M(kx) \leq RM(x)$ for all $x \in (0, r]$. By (16) there exists a positive integer $n_1$ such that

$$M\left(\frac{\nabla \lambda, x_i}{\rho}\right) < \frac{1}{2} r p\left(\frac{r}{2}\right)$$
for all \( i \geq n_1 \). We claim that \( \frac{\nabla \lambda_i}{\rho} \leq r \) for all \( i \geq n_1 \). Otherwise, we can find \( j > n_1 \) with \( \frac{\nabla \lambda_j}{\rho} > r \), and thus

\[
M \left( \frac{\nabla \lambda_i}{\rho} \right) \geq \int_{\rho/2}^{\rho} \rho(t) dt > \frac{1}{2} r \rho \left( \frac{r}{2} \right)
\]

Is a contradiction. Hence our claim is true. Then we can find that

\[
\sum_{i=n_1}^{\infty} M \left( \frac{\nabla \lambda_i}{l} \right) \leq \sum_{i=n_1}^{\infty} M \left( \frac{\nabla \lambda_i}{\rho} \right),
\]

and hence

\[
\sum_{i=1}^{\infty} M \left( \frac{\nabla \lambda_i}{l} \right) < \infty
\]

for every \( l > 0 \). This completes our proof.

**Proposition 2.16.** Let \( M_1 \) and \( M_2 \) be two Orlicz functions. If \( M_1 \) and \( M_2 \) are equivalent then \( \ell_{M_1} (\nabla, \Lambda) = \ell_{M_2} (\nabla, \Lambda) \) and the identity map

\[
I: \left( \ell_{M_1} (\nabla, \Lambda), \| \cdot \|_{M_1} \right) \rightarrow \left( \ell_{M_2} (\nabla, \Lambda), \| \cdot \|_{M_2} \right)
\]

is a topological isomorphism.

**Proof.** Let \( M_1 \) and \( M_2 \) are equivalent and so satisfy (4). Suppose \( x \in \ell_{M_2} (\nabla, \Lambda) \), then

\[
\sum_{i=1}^{\infty} M_2 \left( \frac{\nabla \lambda_i}{l} \right) < \infty
\]

for some \( \rho > 0 \). Hence for some \( l \geq 1 \), \( \frac{\nabla \lambda_i}{lp} \leq x_0 \) for all \( i \geq 1 \). Therefore,

\[
\sum_{i=1}^{\infty} M_1 \left( \frac{\alpha \nabla \lambda_i}{lp} \right) \leq \sum_{i=1}^{\infty} M_2 \left( \frac{\nabla \lambda_i}{\rho} \right) < \infty.
\]

Thus \( \ell_{M_2} (\nabla, \Lambda) \subset \ell_{M_1} (\nabla, \Lambda) \). Similarly \( \ell_{M_1} (\nabla, \Lambda) \subset \ell_{M_2} (\nabla, \Lambda) \). Let us abbreviate here \( \| \cdot \|_{M_1}^{\nabla} \) and \( \| \cdot \|_{M_2}^{\nabla} \) by \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \), respectively. For \( x \in \ell_{M_2} (\nabla, \Lambda) \),

\[
\sum_{i=1}^{\infty} M_2 \left( \frac{\nabla \lambda_i}{\| x \|_2} \right) \leq 1.
\]

One can find \( \mu > 1 \) with \( \frac{x_0}{2} \mu p_2 \left( \frac{x_0}{2} \right) \geq 1 \), where \( p_2 \) is the kernel associated with \( M_2 \). Hence

\[
M_2 \left( \frac{\nabla \lambda_i}{\| x \|_2} \right) \leq \left( \frac{x_0}{2} \right) \mu p_2 \left( \frac{x_0}{2} \right)
\]
for all $i \geq 1$. This implies that $\frac{|\nabla \lambda_i x_i|}{\mu \|x\|_2} \leq x_0$ for all $i \geq 1$. Therefore
$$\sum_{i=1}^{\infty} M_i \left( \frac{\alpha |\nabla \lambda_i x_i|}{\mu \|x\|_2} \right) < 1$$
and so $\|x\| \leq \left( \frac{\mu}{\alpha} \right) \|x\|_2$. Similarly we can show $\|x\| \leq \beta \|x\|$ by choosing $\gamma$ with $\gamma \beta > 1$ such that $\gamma \beta \left( \frac{x_0}{2} \right) p_i \left( \frac{x_0}{2} \right) \geq 1$. Thus $\alpha \mu^{-1} \|x\| \leq \|x\|_2 \leq \beta \|x\|$ which establishes that $I$ is a topological isomorphism.

**Proposition 2.17.** (i) $\ell_M(\Lambda) \subset \ell_M(\nabla, \Lambda)$,
(ii) $\ell_M(\Lambda) \subset \ell_M(\Delta, \Lambda)$.

**Proof.** (i) Proof follows from the following inequality:
$$\sum_{i=1}^{\infty} M_i \left( \frac{\nabla \lambda_i x_i}{2 \rho} \right) \leq \frac{1}{2} \sum_{i=1}^{\infty} M_i \left( \frac{\lambda_i x_i}{\rho} \right) + \frac{1}{2} \sum_{i=1}^{\infty} M_i \left( \frac{\lambda_{i-1} x_{i-1}}{\rho} \right),$$

(ii) Proof is similar to that of part (i).

**Proposition 2.18.** Let $M$ be an Orlicz function and $p$ the corresponding kernel. If $p(x) = 0$ for all $x$ in $[0, x_0]$ where $x_0$ is some positive number, then $\ell_M(\nabla, \Lambda)$ is topologically isomorphic to $\ell_\infty(\nabla, \Lambda)$ and $h_M(\nabla, \Lambda)$ is topologically isomorphic to $c_0(\nabla, \Lambda)$.

**Proof.** Let $p(x) = 0$ for all $x$ in $[0, x_0]$. If $y \in \ell_\infty(\nabla, \Lambda)$, then we can find a $\rho > 0$ such that $\frac{|\nabla \lambda_i y_i|}{\rho} \leq x_0$ for $i \geq 1$, and so $\sum_{i=1}^{\infty} M_i \left( \frac{\nabla \lambda_i y_i}{\rho} \right) < \infty$, giving thus $y \in \ell_M(\nabla, \Lambda)$. On the other hand let $y \in \ell_M(\nabla, \Lambda)$, then $\sum_{i=1}^{\infty} M_i \left( \frac{\nabla \lambda_i y_i}{\rho} \right) < \infty$, for some $\rho > 0$ and so $|\nabla \lambda_i y_i| \ll \infty$ for all $i \geq 1$, giving thus $y \in \ell_\infty(\nabla, \Lambda)$. Hence $y \in \ell_\infty(\nabla, \Lambda)$ if and only if $y \in \ell_M(\nabla, \Lambda)$. We can easily find an $x_1$ with $M(x_1) \geq 1$. Let $y \in \ell_\infty(\nabla, \Lambda)$ and $\alpha = \|y\|_\infty = \sup_i (|\nabla \lambda_i y_i|) > 0$. (It is easy to show that $\|y\|_\infty = \sup_i (|\nabla \lambda_i y_i|)$ is a norm on $\ell_\infty(\nabla, \Lambda)$). For every $\varepsilon, 0 < \varepsilon < \alpha$, we can determine $y_j$ with $|\nabla \lambda_j y_j| > \alpha - \varepsilon$ and so
$$\sum_{i=1}^{\infty} M_i \left( \frac{|\nabla \lambda_i y_i|}{\alpha} x_i \right) \geq M_i \left( \frac{(|\nabla \lambda_j y_j| - \alpha + \varepsilon) x_i}{\alpha} \right).$$
Since $M$ is continuous, we find \( \sum_{i=1}^\infty M \left( \frac{\nabla \lambda_i y_i}{\alpha} \right) \geq 1 \), and so \( \|y\| \leq x_i \|y\| \), for otherwise \( \sum_{i=1}^\infty M \left( \frac{\nabla \lambda_i y_i}{\alpha} \right) = 1 \) is a contradiction by Proposition 2.7. Again, \( \sum_{i=1}^\infty M \left( \frac{\nabla \lambda_i y_i}{\alpha} \right) = 0 \) and it follows that \( \|y\| \leq \frac{1}{x_0} \|y\|_\infty \). Thus the identity map

\[
I: \left( \ell_M (\nabla, \Lambda), \|\| \right) \rightarrow \left( \ell_\infty (\nabla, \Lambda), \|\| \right)
\]

is a topological isomorphism.

For the last part, let \( y \in h_M (\nabla, \Lambda) \), then for any \( \varepsilon > 0, |\nabla \lambda_i y_i| \leq \varepsilon x_i \), for all sufficiently large \( i \), where \( x_i \) is some positive number with \( p(x_i) > 0 \). Hence \( y \in c_0 (\nabla, \Lambda) \). Next let \( y \in c_0 (\nabla, \Lambda) \). Then for any \( \rho > 0, \|\nabla \lambda_i y_i / \rho \| < \frac{1}{2} x_0 \) for all sufficiently large \( i \). Thus \( M \left( \frac{\nabla \lambda_i y_i}{\rho} \right) \to \infty \) for all \( \rho > 0 \) and so \( y \in h_M (\nabla, \Lambda) \). Hence \( h_M (\nabla, \Lambda) = c_0 (\nabla, \Lambda) \) and we are done.

**Corollary 2.19.** Let \( M \) be an Orlicz function and \( p \) the corresponding kernel. If \( p(x) = 0 \) for all \( x \) in \( [0, x_0] \) where \( x_0 \) is some positive number, then \( \ell_M (\nabla, \Lambda) \) is topologically isomorphic to \( \ell_\infty \) and \( h_M (\nabla, \Lambda) \) is topologically isomorphic to \( c_0 \).

**Proof.** Let us define the mapping for \( Z = \ell_\infty, c_0 \)

\[
T: Z (\nabla, \Lambda) \rightarrow Z
\]

by \( Tx = (\nabla \lambda_i x_i) \), for every \( x \in Z (\nabla, \Lambda) \). Then clearly \( T \) is a linear homeomorphism.

Hence the proof follows from Proposition 2.18.

**Lemma 2.20.** Let \( M \) be an Orlicz function. Then \( x \in \ell_M (\Delta, \Lambda) \) implies \( \{ k^{-1} \Delta^k x_i \} \in \ell_\infty \).

**Proof.** Let \( x \in \ell_M (\Delta, \Lambda) \). Then, one can easily prove that \( \{ \Delta^k x_i \} \in \ell_\infty \) which gives the result \( \{ k^{-1} \Delta^k x_i \} \in \ell_\infty \).

**Proposition 2.21.** Let \( M \) be an Orlicz function and \( p \) be the corresponding kernel of \( M \). If \( p(x) = 0 \) for all \( x \) in \( [0, x_0] \), where \( x_0 \) is some positive number, then

(i) Köthe-Toeplitz dual of \( \ell_M (\Delta, \Lambda) \) is \( D_1 \), where

\[
D_1 = \left\{ (a_k): \sum_{k=1}^\infty k \left| \Delta^k a_k \right| < \infty \right\}.
\]
(ii) Köthe-Toeplitz dual of $D_1$ is $D_2$, where
\[ D_2 = \left\{ (b_k) : \sup_k k^{-1} |\lambda_k b_k| < \infty \right\}. \]

Proof. (i) Let $a \in D_1$ and $x \in \ell_M(\Delta, \Lambda)$. Then
\[ \sum_{k=1}^{\infty} |a_k x_k| = \sum_{k=1}^{\infty} k \lambda_k^{-1} a_k k^{-1} |\lambda_k x_k| \leq \sup_k k^{-1} |\lambda_k x_k| \sum_{k=1}^{\infty} k \lambda_k^{-1} a_k < \infty. \]

Hence $a \in \left[ \ell_M(\Delta, \Lambda) \right]^\alpha$. Thus, the inclusion $D_1 \subset \left[ \ell_M(\Delta, \Lambda) \right]^\alpha$ holds.

Conversely suppose that $a \in \left[ \ell_M(\Delta, \Lambda) \right]^\alpha$. Then $\sum_{k=1}^{\infty} |a_k x_k| < \infty$ for every $x \in \ell_M(\Delta, \Lambda)$. So we can take $x_k = \lambda_k^{-1} k$ for all $k \geq 1$, because then $(x_k) \in \ell_M(\Delta, \Lambda)$ and hence $(x_k) \in \ell_M(\Delta, \Lambda)$ as shown in Proposition 2.18.

Now $\sum_{k=1}^{\infty} k \lambda_k^{-1} a_k = \sum_{k=1}^{\infty} |a_k x_k| < \infty$ and thus $a \in D_1$. Hence, the inclusion $\left[ \ell_M(\Delta, \Lambda) \right]^\alpha \subset D_1$ holds.

(ii) Proof follows by similar arguments used in the prove of case (i).

Proposition 2.22. Let $M$ be an Orlicz function and $p$ be the corresponding kernel of $M$. If $p(x) = 0$ for all $x$ in $[0, x_0]$, where $x_0$ is some positive number, then Köthe-Toeplitz dual of $h_M(\Delta, \Lambda)$ is $D_1$, where $D_1$ is defined as in Proposition 2.21.

Proof. Let $a \in D_1$ and $x \in h_M(\Delta, \Lambda)$. Then
\[ \sum_{k=1}^{\infty} |a_k x_k| = \sum_{k=1}^{\infty} k \lambda_k^{-1} a_k k^{-1} |\lambda_k x_k| \leq \sup_k k^{-1} |\lambda_k x_k| \sum_{k=1}^{\infty} k \lambda_k^{-1} a_k < \infty. \]

Hence $a \in \left[ h_M(\Delta, \Lambda) \right]^\alpha$, that is the inclusion $D_1 \subset \left[ h_M(\Delta, \Lambda) \right]^\alpha$ holds.

Conversely suppose that $a \in \left[ h_M(\Delta, \Lambda) \right]^\alpha$ and $a \notin D_1$. Then there exists a strictly increasing sequence $(n_i)$ of positive integers such that $n_1 < n_2 < \ldots$, such that
\[ \sum_{k=n_{i-1}+1}^{n_i} \lambda_k^{-1} k |a_k| > i. \]

Define $(x_k)$ by
\[ x_k = \begin{cases} 0 & , \quad 1 \leq k \leq n_1 \\ k \lambda_k^{-1} \text{sgn } a_k / i & , \quad n_j < k \leq n_{i+1} \end{cases} \]

Then $(x_k) \in c_0(\Delta, \Lambda)$ and so by Proposition 2.18, $(x_k) \in h_M(\Delta, \Lambda)$. Then we have
\[ \sum_{k=1}^{\infty} |a_kx_k| = \sum_{k=n+1}^{\infty} |a_kx_k| + \ldots + \sum_{k=n+1}^{\infty} |a_kx_k| + \ldots \]
\[ = \sum_{k=n+1}^{\infty} k |\lambda^{-1} a_k| + \ldots + \frac{1}{i} \sum_{k=n+1}^{\infty} k |\lambda^{-1} a_i| + \ldots > 1 + 1 + \ldots = \infty. \]

This contradicts to \( a \in \left[ h_M(\Delta, \Lambda) \right]^\alpha \). Hence \( a \in D_1 \), i.e. the inclusion \( \left[ h_M(\Delta, \Lambda) \right]^\alpha \subset D_1 \) also holds. This completes the proof.

References

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